Branes, Pseudoforms and Lie Superalgebra Cohomology

Carlo Alberto Cremonini

Charles University, Prague

May 16th, 2022 Bayrischzell

Joint work with P.A.Grassi and B.Jurčo.

Outline

- Motivations
- Lie Algebra Cohomology (Chevalley-Eilenberg, Koszul, Hochschild-Serre, etc.)
- Forms on Supermanifolds (Berezin, Bernstein, Leites, Manin, Penkov, Witten, etc.)
- Lie Superalgebra Cohomology: superforms (Kac, Fuks, etc.)
- Lie Superalgebra Cohomology: integral and pseudoforms

Motivations: Branes and Supergravity

- Brane Scan (Achucarro-Evans-Townsend-Wiltshire, Green-Schwarz-Witten, Duff, etc.)
- Free Differential Algebras (Sullivan, Castellani-d'Auria-Fré): given $\omega \in H^n(\mathfrak{g})$, we introduce a new generator η^{n-1} s.t.

$$d\eta^{i(n-1)} = \omega^{i(n)} .$$

  Sullivan's theorems: - Every free differential algebra is isomorphic to the tensor product of a unique minimal algebra and a unique contractible algebra.
  -a minimal differential algebra is determined by the dual Lie algebra defined by the quadratic equations and its cohomology classes that determine the differential algebra extension.
- L_∞-algebras: given $\omega^{i(n)} = C^i_{i_1 \ldots i_n} V^{i_1} \wedge \ldots \wedge V^{i_n}$, then

$$d\eta^{i(n-1)} = \omega^{i(n)} \iff [X_{i_1}, \ldots, X_{i_n}] = C^i_{i_1 \ldots i_n} \tilde{\eta}_i .$$

- Supergravity multiplets, e.g., flat $d = 6, N = (2,0)$ (or (4,0))

$$\left(V^a, \psi^A_\alpha, \omega^{ab}, B^{AB} \right)$$

B comes from the 3−cocylce.

- Branes and higher WZW models (Fiorenza, Sati, Schreiber):

$$d\eta^{(n-1)} = \omega^{(n)} \ , \ \exp \left(i \int_{\Sigma^{n-1}} \mathcal{L}_{WZW} \right) = \exp \left(i \int_{\Sigma^{n-1}} \eta^{(n-1)} \right) \ .$$

- Goal: extend to unexplored complexes, pseudoforms.

$$d\eta^{(n-1|q)} = \omega^{(n|q)} \ \overset{?}{\Longrightarrow} \ \Sigma^{(p-1|q)} \ .$$

Superbranes? New supergravity couplings?

Introducing Lie Algebra Cohomology

Let $\mathfrak{g}$ be an finite dimensional Lie (suer)algebra defined over the field $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and let V be a $\mathfrak{g}$-module; a *p-chain* of $\mathfrak{g}$ valued in V is an (graded)alternating $\mathbb{K}$-linear map

$$C_p(\mathfrak{g}, V) := \wedge^p \mathfrak{g} \otimes V \ ,$$

where $\wedge^p \mathfrak{g}$ is for $\mathfrak{g}$ considered a vector space. This can be lifted to a complex by introducing the differential $\partial : C_p(\mathfrak{g}, V) \to C_{p-1}(\mathfrak{g}, V)$

$$\partial \left[f \otimes \left(\mathcal{Y}_{a_1} \wedge \ldots \wedge \mathcal{Y}_{a_p} \right) \right] = \sum_{i=1}^{p} (-1)^{\delta_i} \left(\mathcal{Y}_{a_i} f \right) \otimes \left(\mathcal{Y}_{a_1} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_i} \wedge \ldots \wedge \mathcal{Y}_{a_p} \right) + \\ + \sum_{i<j}^{p} (-1)^{\delta_{i,j}} f \otimes \left(\left[\mathcal{Y}_{a_i}, \mathcal{Y}_{a_j} \right] \wedge \mathcal{Y}_{a_1} \wedge \ldots \wedge \hat{\mathcal{Y}}_{a_i} \wedge \hat{\mathcal{Y}}_{a_j} \wedge \ldots \wedge \mathcal{Y}_{a_p} \right) \ .$$

On the dual side, we can define *p-cochains* as

$$C^{p}\left(\mathfrak{g},V\right):=Hom_{\mathbb{K}}\left(\wedge^{p}\mathfrak{g},V\right)=\wedge^{p}\mathfrak{g}^{*}\otimes V=\bigoplus_{r=1}^{p}\left(\wedge^{r}\mathfrak{g}_{0}^{*}\otimes S^{q-r}\mathfrak{g}_{1}^{*}\right)\otimes V\ .$$

Again, we can introduce a differential $d : C^{p}\left(\mathfrak{g},V\right)\to C^{p+1}\left(\mathfrak{g},V\right)$

$$\begin{aligned} d\omega\left(\mathcal{Y}_{a_{1}},\ldots,\mathcal{Y}_{a_{p+1}}\right)=&\sum_{i=1}^{p+1}(-1)^{\delta_{i}}\mathcal{Y}_{a_{i}}\omega\left(\mathcal{Y}_{a_{1}}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_{i}}\wedge\ldots\wedge\mathcal{Y}_{a_{p+1}}\right)+\\ &+\sum_{i<j}(-1)^{\delta_{i,j}}\omega\left(\left[\mathcal{Y}_{a_{i}},\mathcal{Y}_{a_{j}}\right]\wedge\mathcal{Y}_{a_{1}}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_{i}}\wedge\ldots\wedge\hat{\mathcal{Y}}_{a_{j}}\wedge\ldots\wedge\mathcal{Y}_{a_{p+1}}\right)\ .\end{aligned}$$

$$d^{2}=0\iff(graded)Jacobi$$

Relative Lie (super)algebra cohomology

Given a Lie (super)algebra $\mathfrak{g}$ and a Lie sub-(super)algebra $\mathfrak{h}$ (we denote $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$), we define the space of *horizontal p-cochains* with values in a module V as

$$C^p(\mathfrak{k}, V) := \{\omega \in C^p(\mathfrak{g}, V) : \iota_{\mathcal{Y}_a}\omega = 0, \forall \mathcal{Y}_a \in \mathfrak{h}\} \ ,$$

We define the space of $\mathfrak{h}$*-invariant p-cochains* with values in V as

$$\left(C^p(\mathfrak{g}, V)\right)^{\mathfrak{h}} := \{\omega \in C^p(\mathfrak{g}, V) : \mathcal{L}_{\mathcal{Y}_a}\omega = 0, \forall \mathcal{Y}_a \in \mathfrak{h}\} \ .$$

Forms which are both horizontal and $\mathfrak{h}$-invariant are called *basic*:

$$\left(C^p(\mathfrak{k}, V)\right)^{\mathfrak{h}} \equiv \left(C^p(\mathfrak{g}/\mathfrak{h}, V)\right)^{\mathfrak{h}} \ .$$

We define the *cohomology of* $\mathfrak{g}$ *relative to* $\mathfrak{h}$ as

$$H^\bullet(\mathfrak{g}, \mathfrak{h}, V) := \frac{\left\{\omega \in \left(C^\bullet(\mathfrak{k}, V)\right)^{\mathfrak{h}} : \nabla_{\mathfrak{k}}\omega = 0\right\}}{\left\{\omega \in \left(C^\bullet(\mathfrak{k}, V)\right)^{\mathfrak{h}} : \exists \eta \in \left(C^\bullet(\mathfrak{k}, V)\right)^{\mathfrak{h}}, \omega = \nabla_{\mathfrak{k}}\eta\right\}} \ ,$$

where the differential is

$$d\big|_{basic} \equiv \nabla_{\mathfrak{k}} \ .$$

Forms on Supermanifolds

Let $\mathcal{SM} = \left(|\mathcal{SM}| \, , C^{\infty}_{\mathbb{R}^m} \left[\theta^1, \ldots, \theta^n \right] \right)$ of dimension $dim\mathcal{SM} = (m|n)$ be a (smooth) supermanifold.

- complex of *superforms*: $\left(\Omega^{(\bullet|0)} \left(\mathcal{SM}, d_{dR} \right) \right)$. It is *unbounded from above*:

$$0 \xrightarrow{\mathrm{d}} \Omega^{(0|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \Omega^{(1|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \ldots \xrightarrow{\mathrm{d}_{dR}} \Omega^{(m|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}_{dR}} \ldots$$

  as a consequence of the commutation relations

$$dx^i \wedge dx^j = -dx^j \wedge dx^i \ , \ dx^i \wedge d\theta^\alpha = d\theta^\alpha \wedge dx^i \ , \ d\theta^\alpha \wedge d\theta^\beta = d\theta^\beta \wedge d\theta^\alpha \ .$$

The notion of *top form* has to be found in the *Berezinian line bundle*, the super-analogous of the *Determinant line bundle*. With the Berezinian bundle one defines the complex of *integral forms*, which is unbounded from below:

$$\ldots \xrightarrow{\delta} \Omega^{(0|n)}_{\mathcal{SM}} \xrightarrow{\delta} \ldots \xrightarrow{\delta} \Omega^{(m-1|n)}_{\mathcal{SM}} \xrightarrow{\delta} \Omega^{(m|n)}_{\mathcal{SM}} \xrightarrow{\delta} 0 \ ,$$

where we denoted $\Omega^{(p|n)}_{\mathcal{SM}} := \mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$.

Berezinian: Polyvector Fields Realisation

The Berezinian is defined as

$$\mathcal{B}er\left(\mathcal{SM}\right) := \left(\mathcal{B}er\Omega^1_{odd}\left(\mathcal{SM}\right)\right)^* \ .$$

Let $x^i|\theta^\alpha$, $i = 1, \ldots, n$ and $\alpha = 1, \ldots, m$ be local coordinates, then

$$\mathcal{B}er\left(\mathcal{SM}\right) \cong \mathcal{O} \cdot \left[\bigwedge_{i=1}^{\dim \mathcal{SM}_0} dx^i \otimes \bigwedge_{\alpha=1}^{\dim \mathcal{SM}_1} \partial_{\theta^\alpha} \right] \ .$$

The differential δ is defined via Lie derivative:

$$\delta = \sum_a \mathcal{L}^R_{\frac{\partial}{\partial z^a}} \otimes_{\mathbb{K}} \frac{\partial}{\partial\left(\pi \frac{\partial}{\partial z^a}\right)} (-1)^{|z^a|}$$

Let $\mathcal{Y}_A = \{X_i, \chi_\alpha\}$ and $\mathcal{Y}^{*A} = \left\{V^i, \psi^\alpha\right\}$ the bases of vectors and MC forms, respectively, the *Haar Berezinian* reads

$$\mathcal{B}er\left(\mathfrak{g}\right) := V \cdot \left[\bigwedge_{i=1}^{m} V^i \otimes \bigwedge_{\alpha=1}^{n} \xi_\alpha \right] \equiv V \cdot \mathcal{D} \ .$$

Integral forms are then defined as

$$C^p_{int}(\mathfrak{g}) := \mathcal{B}er(\mathfrak{g}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathfrak{g})) \ ,$$

$$\delta\left(\mathcal{D} \otimes \left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right]\right) = \mathcal{D} \otimes \partial\left[f \otimes \mathcal{Y}_1 \wedge \ldots \wedge \mathcal{Y}_{m-p}\right] \ .$$

Berezinian: Distributional Realisation

One way to realise integral forms is as (compactly supported) *generalised functions* on $\mathrm{Tot}\,\Pi\mathcal{T}^*(\mathcal{SM})$, that is elements

$$\omega(x^1, \dots, x^n, d\theta^1, \dots, d\theta^m | \theta^1, \dots, \theta^m, dx^1, \dots dx^n) \in \Pi\mathcal{T}(\mathscr{M}) \ ,$$

where $x^i|\theta^\alpha$ are local coordinates for $\mathcal{SM}$, which only allow a *distributional dependence* supported in $d\theta^1 = \ldots = d\theta^m = 0$. $(\bullet|n)$-integral forms are defined as

$$\mathscr{B}er\left(\mathcal{SM}\right) \otimes S^{n-p}(\Pi\mathcal{T}(\mathfrak{g})) \coloneqq \mathcal{O} \cdot (\iota_{\mathcal{Y}})^{n-p} \cdot \left[\bigwedge_{i=1}^{\dim \mathcal{SM}_0} dx^i \wedge \bigwedge_{\alpha=1}^{\dim \mathcal{SM}_1} \delta\left(d\theta^\alpha\right) \right]$$

and it locally reads

$$\omega^{(p|n)} = \omega_{[i_1 \dots i_q j_1 \dots j_n]}\left(x, \theta\right) dx^{i_1} \wedge \dots dx^{i_q} \wedge \delta^{(j_1)}\left(d\theta^1\right) \wedge \ldots \wedge \delta^{(j_n)}\left(d\theta^n\right) \ .$$

We are denoting $\delta^{(j)}\left(d\theta^\alpha\right) \coloneqq \left(\iota_\alpha\right)^j \delta\left(d\theta^\alpha\right)$. These distribution satisfy the relations

$$\left|\delta^{(j)}\left(d\theta^\alpha\right)\right| = 1 \ , \ \forall j \in \mathbb{N} \cup \{0\} \ , \ \delta\left(d\theta^\alpha\right) \wedge \delta\left(d\theta^\beta\right) = -\delta\left(d\theta^\beta\right) \wedge \delta\left(d\theta^\alpha\right) \ ,$$

$$d\theta^\alpha \delta^{(j)}\left(d\theta^\alpha\right) = -j\delta^{(j-1)}\left(d\theta^\alpha\right) \ , \ \delta\left(\lambda d\theta^\alpha\right) = \frac{1}{\lambda}\delta\left(d\theta^\alpha\right) \ .$$

The differential δ is defined as $\delta = d_{dR}$, acting on integral forms via the previous formal relations.
The *Haar Berezinian* is now defined as

$$\mathscr{B}er\left(\mathfrak{g}\right) := V \cdot \left[\bigwedge_{i=1}^{m} V^i \otimes \bigwedge_{\alpha=1}^{n} \delta\left(\psi^\alpha\right)\right] \equiv V \cdot \mathcal{D} \equiv \omega_{\mathfrak{g}}^{top} \ .$$

Any integral form can be obtained by acting on $\omega_{\mathfrak{g}}^{top}$ with contractions:

$$\omega^{(m-p|n)} = \iota_{\mathscr{Y}_{A_1}} \dots \iota_{\mathscr{Y}_{A_p}} \omega_{\mathfrak{g}}^{top} \ ,$$

thus reproducing the structure $\mathcal{B}er(\mathcal{SM}) \otimes S^{n-p}(\Pi\mathcal{T}(\mathcal{SM}))$.

Dictionary: Dirac $\leftrightsquigarrow$ Koszul

In both realisations we have that integral forms are defined as

$$C^p_{int}(\mathfrak{g}, V) := \mathcal{B}er(\mathfrak{g}, V) \otimes S^p \Pi \mathfrak{g} \ .$$

We have

$$\begin{aligned} \mathcal{B}er : V \cdot \left(\bigwedge_{i=1}^m \mathcal{V}^i \right) \left(\bigwedge_{\alpha=1}^m \delta(\psi^\alpha) \right) \quad &\leftrightsquigarrow \quad V \cdot \left[\bigwedge_{i=1}^m \mathcal{V}^i \otimes \bigwedge_{\alpha=1}^m \chi_\alpha \right] \\ \omega^{m-p}_{int} \left(\equiv \omega^{(m-p|n)} \right) : \left(\prod_{i=1}^p \iota_{\mathcal{Y}_i} \right) \mathcal{B}er \quad &\leftrightsquigarrow \quad \mathcal{B}er \otimes \left(\bigwedge_{i=1}^p \pi \mathcal{Y}^i \right) \\ d_{CE} : d \quad &\leftrightsquigarrow \quad \delta = 1 \otimes \partial \end{aligned}$$

Distributional Realisation: Pseudoforms

The distributional realisation suggests the construction of a different type of forms, with *non-maximal* and *non-zero* number of delta's:

$$\omega^{(p|s)} = \omega_{[i_1 \ldots i_q j_1 \ldots j_n]} (x, \theta) \, dx^{i_1} \wedge \ldots dx^{i_q} \wedge \delta^{(j_1)} \left(d\theta^1 \right) \wedge \ldots \wedge \delta^{(j_s)} \left(d\theta^s \right) \ , \ 0 < s < n \ .$$

These objects are not well defined; for example, they do not behave tensorially

$$d\theta^\alpha \mapsto \Lambda^\alpha_\mu dx^\mu + \Lambda^\alpha_\beta d\theta^\beta \ , \ \delta \left(d\theta^\alpha \right) \mapsto \delta \left(\Lambda^\alpha_\mu dx^\mu + \Lambda^\alpha_\beta d\theta^\beta \right) = \ldots ?$$

Example

$$d\theta^1 \mapsto d\theta^1 + d\theta^2 \implies \delta \left(d\theta^1 \right) \mapsto \delta \left(d\theta^1 + d\theta^2 \right) = \sum_{i=0}^{\infty} \frac{\left(d\theta^2 \right)^i}{i!} \delta^{(i)} \left(d\theta^1 \right) \ .$$

Some hints to define pseudoforms: Manin, Witten.

Pseudoforms as Infinite-Dimensional Representations

Given $\mathfrak{g}$, $\dim \mathfrak{g} = (m|n)$, $\mathfrak{h}$, $\dim \mathfrak{h} = (p|q)$ and $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$, $\dim \mathfrak{k} = (m-p|n-q)$, we can define

$$\mathcal{B}er\left(\mathfrak{h}\right) := \mathbb{K} \cdot \left[\bigwedge_{i=1}^{p} V^i \wedge \bigwedge_{\alpha=1}^{q} \delta\left(\psi^\alpha\right) \right] \ , \ V^i, \psi^\alpha \in \Pi\mathfrak{h}^* \ , \ \xi_\alpha \in \mathfrak{h} \ ,$$

$$\mathcal{B}er\left(\mathfrak{k}\right) := \mathbb{K} \cdot \left[\bigwedge_{\hat{i}=1}^{m-p} V^{\hat{i}} \wedge \bigwedge_{\hat{\alpha}=1}^{n-q} \delta\left(\psi^{\hat{\alpha}}\right) \right] \ , \ V^{\hat{i}}, \psi^{\hat{\alpha}} \in \Pi\mathfrak{k}^* \ , \ \xi_{\hat{\alpha}} \in \mathfrak{k} \ .$$

They are not $\mathfrak{g}$-modules. We can use them to construct $\mathfrak{g}$-modules:

$$\begin{aligned} V_{\mathfrak{h}}^{(p|q)} &:= \bigoplus_{i=0}^{\infty} \left(S^i \Pi\mathfrak{h} \otimes \mathcal{B}er\left(\mathfrak{h}\right) \right) \otimes S^i \Pi\mathfrak{k}^* = \bigoplus_{i=0}^{\infty} C_{int}^{m-i}\left(\mathfrak{h}\right) \otimes C^i\left(\mathfrak{k}\right) \ , \\ V_{\mathfrak{k}}^{(m-p|n-q)} &:= \bigoplus_{i=0}^{\infty} \left(S^i \Pi\mathfrak{k} \otimes \mathcal{B}er\left(\mathfrak{k}\right) \right) \otimes S^i \Pi\mathfrak{h}^* = \bigoplus_{i=0}^{\infty} C_{int}^{m-p-i}\left(\mathfrak{k}\right) \otimes C^i\left(\mathfrak{h}\right) \ . \end{aligned}$$

Then pseudoforms are

$$C^{p\pm s}\left(\mathfrak{g}, V_{\mathfrak{h}}^{(p|q)}\right) \equiv C^{(p\pm s|q)}(\mathfrak{g}) \ , \ C^{m-p\pm s}\left(\mathfrak{g}, V_{\mathfrak{k}}^{(m-p|n-q)}\right) \equiv C^{(m-p\pm s|n-q)}(\mathfrak{g}) \ .$$

They can be naturally equipped with a differential constructed from d and ∂. In the distributional realisation we have (consider $V = \mathbb{K}$)

$$\begin{aligned} d : C^{(s|\bullet)}(\mathfrak{g}) &\quad \rightarrow \quad C^{(s+1|\bullet)}(\mathfrak{g}) \\ \omega &\quad \mapsto \quad d\omega = \sum_{a,b,c} f^a_{bc} \mathcal{Y}^{*b} \wedge \mathcal{Y}^{*c} \wedge \iota_{\mathcal{Y}_c}\omega \ . \end{aligned}$$

The pseudoform cohomology is defined as

$$\begin{aligned} H^{(\bullet|q)}(\mathfrak{g}) &\quad := \quad H^{\bullet}\left(\mathfrak{g}, V_{\mathfrak{h}}^{(p|q)}\right) \ , \\ H^{(\bullet|n-q)}(\mathfrak{g}) &\quad := \quad H^{\bullet}\left(\mathfrak{g}, V_{\mathfrak{k}}^{(m-p|n-q)}\right) \ . \end{aligned}$$

Fuks' Theorems

Theorem

If $m \geq n$, the natural inclusions

$$\begin{aligned} \mathfrak{gl}(m) &\to \mathfrak{gl}(m|n)_0 &\subset \mathfrak{gl}(m|n) \ , \\ \mathfrak{sl}(m) &\to \mathfrak{sl}(m|n)_0 &\subset \mathfrak{sl}(m|n) \ , \end{aligned}$$

induce an isomorphism in cohomology with trivial coefficients.

Theorem

$$H^\bullet(\mathfrak{osp}(m|n)) = \begin{cases} H^\bullet(\mathfrak{so}(m)) \ , \ \textit{if } m \geq 2n \ , \\ H^\bullet(\mathfrak{sp}(n)) \ , \ \textit{if } m < 2n \ . \end{cases}$$

RMK

Only a part of the bosonic subalgebra contributes to the CE cohomology. The CE cohomology is related to the superalgebra invariants (or analogously to its rank): it looks like "some invariants get lost".

The Berezinian Complement Isomorphism

We define the "Berezinian complement" map $\star$ as

$$\begin{aligned} \star : C^p(\mathfrak{g}) &\rightarrow C^{m-p}_{int}(\mathfrak{g}) \\ \omega &\mapsto \star\omega^{(p|0)} = (\star\omega)^{(m-p|n)} := \left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}} \right) \omega^{top}_{\mathfrak{g}} \ , \end{aligned}$$

where $\left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}} \right) \omega = 1$. This map induces a cohomology isomorphism:

$$\star : H^\bullet(\mathfrak{g}) \xrightarrow{\cong} H^{m-\bullet}_{int}(\mathfrak{g}) \ .$$

The isomorphism is verified when $\mathfrak{g}$ admits non-degenerate invariant bilinear form, hence it holds e.g. for "basic Lie superalgebras".

Spectral Sequences

The idea is to reconstruct the cohomology of a Lie algebra starting from the (eventually known) cohomology of substructure.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \ , \ \mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \mathfrak{h} \text{ sub-algebra.}$$

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ .$$

The cohomology is calculated via *approximations*: we can split the differential d as

$$d = d_0 + d_1 + d_2 + \dots$$

then, calculate the cohomology w.r.t. d_0, then d_1, then d_2 etc., up to *convergence*.

- Trivial modules: Koszul
- General modules: Hochschild-Serre

Given a Lie algebra $\mathfrak{g}$ and a Lie sub-algebra $\mathfrak{h}$ (we denote $\mathfrak{g}/\mathfrak{h} = \mathfrak{k}$), we define the filtration

$$F^p C^q(\mathfrak{g}) = \left\{ \omega \in C^q(\mathfrak{g}) : \forall \xi_i \in \mathfrak{h}, \iota_{\xi_{i_1}} \ldots \iota_{\xi_{i_{q+1-p}}} \omega = 0 \right\} ,$$

which is a filtration in the sense that

$$dF^p C^q(\mathfrak{g}) \subseteq F^p C^{q+1}(\mathfrak{g}) \ , \ \forall p, q \in \mathbb{Z} .$$

There exists a spectral sequence $(E_s^{\bullet,\bullet}, d_s)$, $d_s : E_s^{p,q} \to E_s^{p+s,q+1-s}$ that converges to $H(\mathfrak{g})$. The first space (page zero) reads

$$E_0^{m,n} := F^m C^{(m+n)}(\mathfrak{g}) / F^{m+1} C^{(m+n)}(\mathfrak{g}) \ .$$

The differentials d_s are induced by the CE differential:

$$d = V_{\mathfrak{h}} V_{\mathfrak{h}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{h}} + V_{\mathfrak{k}} V_{\mathfrak{k}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} + V_{\mathfrak{k}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} \ ,$$

reflecting the structure

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h} + \mathfrak{k} \ .$$

The following pages of the spectral sequence are defined as

$$E_s^{\bullet,\bullet} := H\left(E_{s-1}^{\bullet,\bullet}, d_{s-1}\right) \ .$$

In particular, the first differential formally reads

$$d_0 = V_{\mathfrak{h}} V_{\mathfrak{h}} \iota_{\mathfrak{h}} + V_{\mathfrak{h}} V_{\mathfrak{k}} \iota_{\mathfrak{k}} \ .$$

Theorem (Koszul, Hochschild-Serre)

If $\mathfrak{h}$ is reductive in $\mathfrak{g}$ (i.e., $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$), hence

$$E_1^{m,n} = H^n(\mathfrak{h}) \otimes \left(\Omega^m(\mathfrak{k})\right)^{\mathfrak{h}} \ .$$

$$E_2^{m,n} = H^n(\mathfrak{h}) \otimes H^m(\mathfrak{g}, \mathfrak{h}) \ .$$

Fuks: superform cohomology of classical Lie superalgebras with $\mathfrak{h} = \mathfrak{g}_0 \implies$ no pseudoforms. New objects can be found by repeating the procedure for *sub-superalgebras*. We can introduce two *inequivalent* filtrations

$$F^p C^{(q|l)}(\mathfrak{g}, V) := \left\{ \omega \in C^{(q|l)}(\mathfrak{g}, V) : \forall \mathcal{Y}_a \in \mathfrak{h}, \iota_{\mathcal{Y}_{a_1}} \ldots \iota_{\mathcal{Y}_{a_{q+1-p}}} \omega = 0 \right\} ,$$
$$\tilde{F}^p C^{(q|l)}(\mathfrak{g}, V) := \left\{ \omega \in C^{(q|l)}(\mathfrak{g}, V) : \forall \mathcal{Y}^{*a} \in \mathfrak{h}^*, \mathcal{Y}^{*a_1} \wedge \ldots \wedge \mathcal{Y}^{*a_{q+1-p}} \wedge \omega = 0 \right\} ,$$

with $p, q \in \mathbb{Z}$ and $l \in \{0, \dim \mathfrak{h}_1, \dim \mathfrak{k}_1, \dim \mathfrak{g}_1\}$.

- The two filtrations coincide if $\mathfrak{g}$ is a bosonic Lie algebra;
- if $\mathfrak{h}$ has non-trivial odd part, the second filtration is empty on superforms;
- If $\mathfrak{h}$ has non-trivial odd part, the first filtration is empty on integral forms, which are then kept into account by the second filtration only.

Page zero of the spectral sequence is defined, for any l, as

$$\mathcal{E}_0^{m,n} := E_0^{m,n} \oplus \tilde{E}_0^{m,n} := \frac{F^m \Omega^{(m+n|l)}(\mathfrak{g})}{F^{m+1}\Omega^{(m+n|l)}(\mathfrak{g})} \oplus \frac{\tilde{F}^{m+2n-r}\Omega^{(m+n|l)}(\mathfrak{g})}{\tilde{F}^{m+2n-r+1}\Omega^{(m+n|l)}(\mathfrak{g})} \ .$$

and on pseudoform complexes we obtain

$$l = \dim \mathfrak{h}_1 \quad \implies \quad \mathcal{E}_0^{m,n} = \tilde{E}_0^{m,n} = C^{(m|0)}(\mathfrak{k}) \otimes C^{(n|\dim \mathfrak{h}_1)}(\mathfrak{h}) \ ,$$
$$l = \dim \mathfrak{k}_1 \quad \implies \quad \mathcal{E}_0^{m,n} = E_0^{m,n} = C^{(m|\dim \mathfrak{k}_1)}(\mathfrak{k}) \otimes C^{(n|0)}(\mathfrak{h}) \ .$$

If $\dim \mathfrak{h}_1 = \dim \mathfrak{k}_1 = \dim \mathfrak{g}_1/2$, we have

$$\mathcal{E}_0^{m,n} = \left[C^{(m|\dim \mathfrak{g}_1/2)}(\mathfrak{k}) \otimes C^{(n|0)}(\mathfrak{h}) \right] \oplus \left[C^{(m|0)}(\mathfrak{k}) \otimes C^{(n|\dim \mathfrak{g}_1/2)}(\mathfrak{h}) \right] \ .$$

We construct a spectral sequence

$$d_s : \mathcal{E}_s^{p,q} \to \mathcal{E}_s^{p+s,q+1-s} \ , \ \mathcal{E}_{s+1}^{\bullet,\bullet} := \left(\mathcal{E}_s^{\bullet,\bullet}, d_s \right) \ , \ \mathcal{E}_\infty^{\bullet,\bullet} \cong H^{(\bullet|l)}(\mathfrak{g}, V) \ ,$$

and extend KHS theorem.

Proposition

Let $\mathfrak{g}$ be a Lie (super)algebra over a (characteristic-zero) field $\mathbb{K}$ and let $\mathfrak{h}$ be a Lie sub-(super)algebra reductive in $\mathfrak{g}$, $\dim \mathfrak{h}_1 \neq 0$, and denote $\mathfrak{k} = \mathfrak{g}/\mathfrak{h}$. Then, at picture number $l = \dim \mathfrak{h}_1, \dim \mathfrak{k}_1$, if $\dim \mathfrak{h}_1 \neq \dim \mathfrak{k}_1$, the first pages of the extended spectral sequence read

$$
\begin{aligned}
l = \dim \mathfrak{h}_1 \quad \Longrightarrow \quad & \mathcal{E}_1^{m,n} = \left(C^{(m|0)}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|\dim \mathfrak{h}_{\mathbf{1}})}(\mathfrak{h}) \ , \\
& \mathcal{E}_2^{m,n} = H^{(m|0)}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|\dim \mathfrak{h}_{\mathbf{1}})}(\mathfrak{h}) \ , \\
l = \dim \mathfrak{k}_1 \quad \Longrightarrow \quad & \mathcal{E}_1^{m,n} = \left(C^{(m|\dim \mathfrak{k}_{\mathbf{1}})}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|0)}(\mathfrak{h}) \ , \\
& \mathcal{E}_2^{m,n} = H^{(m|\dim \mathfrak{k}_{\mathbf{1}})}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|0)}(\mathfrak{h}) \ .
\end{aligned}
$$

If $\dim \mathfrak{h}_1 = \dim \mathfrak{k}_1 = \dim \mathfrak{g}_1/2$, then the first two pages at picture number $l = \dim \mathfrak{g}_1/2$ read

$$
\begin{aligned}
\mathcal{E}_1^{m,n} &= \left[\left(C^{(m|0)}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{h}) \right] \oplus \left[\left(C^{(m|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{k}) \right)^{\mathfrak{h}} \otimes H^{(n|0)}(\mathfrak{h}) \right] \ , \\
\mathcal{E}_2^{m,n} &= \left[H^{(m|0)}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{h}) \right] \oplus \left[H^{(m|\dim \mathfrak{g}_{\mathbf{1}}/2)}(\mathfrak{g}, \mathfrak{h}) \otimes H^{(n|0)}(\mathfrak{h}) \right] \ .
\end{aligned}
$$

Example ($\mathfrak{g} = \mathfrak{osp}(2|2)$)

4 even generators, 4 odd generators
From Fuks', we have

$$H^{(\bullet|0)}\left(\mathfrak{osp}(2|2)\right) = H^{\bullet}\left(\mathfrak{sp}(2)\right) = \left\{1, \omega^{(3)}\right\} \ .$$

The abelian factor $\mathfrak{so}(2)$ is the "lost part". We can choose

$$\mathfrak{h} = \mathfrak{osp}(1|2) \ , \ \text{3 bosons, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms;}$$
$$\mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \text{1 boson, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms.}$$

One finds

$$H^{(\bullet|2)}\left(\mathfrak{osp}(2|2)\right) = \left\{\omega^{(0|2)}, \omega^{(1|2)}, \omega^{(3|2)}, \omega^{(4|2)}\right\} \ ,$$

$\omega^{(1|2)}$ encodes the abelian factor.

Example ($\mathfrak{g} = \mathfrak{osp}(1|4)$)

4 even generators, 4 odd generators
From Fuks', we have

$$H^{(\bullet|0)}\left(\mathfrak{osp}(1|4)\right) = H^{\bullet}\left(\mathfrak{sp}(4)\right) = \left\{1, \omega^{(3)}, \omega^{(7)}, \omega^{(10)}\right\} \ .$$

If we choose

$$\mathfrak{h} = \mathfrak{osp}(1|2) \times \mathfrak{sp}(2) \ , \ \text{6 bosons, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms;}$$
$$\mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \text{4 boson, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms.}$$

One finds

$$H^{(\bullet|2)}\left(\mathfrak{osp}(2|2)\right) = \left\{\omega^{(0|2)\otimes 2}, \omega^{(3|2)\otimes 2}, \omega^{(7|2)\otimes 2}, \omega^{(10|2)\otimes 2}\right\} \ .$$

RMK: these classes *are not* $\mathfrak{osp}(1|4)$*-invariant.*

Theorem (Chevalley-Eilenberg)

If $\mathfrak{g}$ is a semi-simple Lie algebra and V a finite-dimensional module, then $H^p(\mathfrak{g}, V) = H^p(\mathfrak{g}, V^{\mathfrak{g}})$.

Theorem (Chevalley-Eilenberg)

If $\mathfrak{g}$ is a semi-simple Lie algebra with values over a characteristic zero field $\mathbb{K}$, every cohomology class $H^q(\mathfrak{g}, \mathbb{K})$ contains a $\mathfrak{g}$-invariant cocycle.

- Extension to superalgebras in the complexes of superforms and integral forms. Failure for pseudoforms: *infinite dimensional representations.*
- There are still some invariant cases, e.g., $\mathfrak{osp}(2|2)$ pseudoforms induced by $\mathfrak{osp}(1|2)$.
- Different choices of sub-superalgebras do not affect superforms and integral forms, but of course pseudoforms: different choices induce different pseudoforms.
- The choice of the sub-superalgebra corresponds to the choice of invariances of the cohomology classes.

Discussion

There are other ways to approach the problem:

- brute force; but pseudoforms live in infinite dimensional spaces, computations may be arbitrarily difficult
- Molien-Weyl integrals: it is possible to extend the bosonic formula to the super setting. Pseudoforms are not standard representations $\rightarrow$ infinite-dimensional representations $\rightarrow$ extremely rich, almost unexplored land

Outlook

- Extension of Sullivan's program to pseudoforms
- Classify cohomology groups among pseudoforms $\overset{?}{\Longrightarrow}$ new branes.
- Partially invariant pseudoforms $\overset{?}{\Longrightarrow}$ broken (super)symmetries/non-linear realisations of (super)symmetries?